Solution by A. H. HOLMES, Brunswick, Me.

$$x^2+w^2+v^2=a^2\text{.........}(1),\quad vw+u(y+z)=bc\text{.........}(4),$$
$$w^2+y^2+u^2=b^2\text{.........}(2),\quad wu+v(z+x)=ca\text{.........}(5),$$
$$v^2+u^2+z^2=a^2\text{.........}(3),\quad uv+w(x+y)=ab\text{.........}(6).$$

From (2), (3), and (4),

$$(w^2+y^2+u^2)(v^2+u^2+z^2)=[vw+u(y+z)]^2.$$
$$\therefore (yv-uw)^2+(uv-zw)^2+(u^2-yz)^2=0.$$
$$\therefore yv=uw\text{.........}(7);\ uv=zw\text{.........}(8);\ u^2=yz\text{.........}(9).$$

From (1), (3), and (5),

$$(x^2+w^2+v^2)(v^2+u^2+z^2)=[wu+v(z+x)]^2.$$
$$\therefore v^2=xz\text{.........}(10);\ vw=xu\text{.........}(11).$$

From (2), (3), and (6),

$$(w^2+y^2+u^2)(v^2+u^2+z^2)=[uv+w(x+y)]^2.$$
$$\therefore w^2=xy\text{.........}(12).$$

Substituting xy and xz for w^2 and v^2 in (1), xy and yz for w^2 and u^2 in (2), and xz and yz for v^2 and u^2 in (3), adding the three equations and extracting square root, we obtain $x+y+z=\sqrt{(a^2+b^2+c^2)}$.

Substituting xu for vw in (4), $u=\dfrac{bc}{\sqrt{(a^2+b^2+c^2)}}$.

Similarly, $v=\dfrac{ac}{\sqrt{(a^2+b^2+c^2)}}$, $w=\dfrac{ab}{\sqrt{(a^2+b^2+c^2)}}$,

$$x=\frac{a^2}{\sqrt{(a^2+b^2+c^2)}},\quad y=\frac{b^2}{\sqrt{(a^2+b^2+c^2)}},\quad z=\frac{c^2}{\sqrt{(a^2+b^2+c^2)}}.$$

227. Proposed by G. I. HOPKINS, A. M., Manchester, N. H.

Solve $x+y+xy+x^2y+xy^2+x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3=11$; $x^4y+3x^3y^2+3x^2y^3+2x^4y^2+4x^2y^3+2x^2y^4+4x^4y^3+4x^3y^4+xy^4+x^5y^2+x^5y^3+2x^4y^4+x^2y^5+x^3y^5=30$.

Solution by F. P. MATZ, Ph. D., Sc. D., Reading, Pa.

Put $X=(x+y+xy+x^2y+xy^2)$, and $Y=(x^3y+2x^2y^2+xy^3+x^3y^2+x^2y^3)$; then the given equations become, respectively,

$$X+Y=11\text{.........}(\alpha),\text{ and}$$
$$XY=30\text{.........}(\beta).$$

$\therefore X=6$, or 5; and $Y=5$, or 6.

By putting $X_1=(x+y+xy)$, and $Y_1=(x^2y+xy^2)$, the expressions represented by X and Y give, respectively,

$$X_1+Y_1=6, \text{ or } 5 \ldots\ldots (\alpha_1), \text{ and}$$
$$X_1Y_1=5, \text{ or } 6 \ldots\ldots (\beta_1).$$

$\therefore X_1=5,\ 1,\ 3$, or 2; and $Y_1=1,\ 5,\ 2$, or 3.

By putting $X_2=(x+y)$ and $Y_2=(xy)$, the expressions represented by X_1 and Y_1 give, respectively,

$$X_2+Y_2=5,\ 1,\ 3, \text{ or } 2 \ldots\ldots (\alpha_2), \text{ and}$$
$$X_2Y_2=1,\ 5,\ 2, \text{ or } 3 \ldots\ldots (\beta_2).$$

$$\therefore x+y=\tfrac{1}{2}(5\pm\sqrt{21}),\ \tfrac{1}{2}(1\pm\sqrt{-19}),\ 2 \text{ or } 1, \text{ or } 1\pm\sqrt{-2}; \text{ and}$$
$$xy=\tfrac{1}{2}(5\mp\sqrt{21}),\ \tfrac{1}{2}(1\mp\sqrt{-19}),\ 1 \text{ or } 2, \text{ or } 1\mp\sqrt{-2}.$$

Solving these *eight* simultaneous equations, we have the *sixteen* values of x and y.

Also solved by J. Scheffer, Henry Heaton, G. B. M. Zerr, and A. H. Holmes.

GEOMETRY.

254. Proposed by W. J. GREENSTREET, M. A., Editor of the Mathematical Gazette, Stroud, England.

Find the cartesian equation to a line that is both tangent and normal to the cardioid.

[No solution of this problem has been received.]

255. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the envelope of the straight line that connects the extremities of two conjugate diameters of an ellipse.

I. Solution by G. W. GREENWOOD, M. A., Professor of Mathematics, McKendree College, Lebanon, Ill.

If we project the ellipse into a circle, the conjugate diameters are projected into perpendicular diameters of the circle, whose chord envelopes a concentric circle. Hence in the original figure the chord envelopes a similar, and similarly situated concentric ellipse.

II. Solution by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

The equation to the line is, with usual notation,

$$\frac{x}{a}\cos(\varphi+\tfrac{1}{4}\pi)+\frac{y}{b}\sin(\varphi+\tfrac{1}{4}\pi)=\cos\tfrac{1}{4}\pi, \text{ or } \left(\frac{x}{a}+\frac{y}{b}\right)\cos\varphi-\left(\frac{x}{a}-\frac{y}{b}\right)\sin\varphi=1,$$

$$i.e., \left(\frac{x}{a}+\frac{y}{b}-1\right)-2\left(\frac{x}{a}-\frac{y}{b}\right)\tan\tfrac{1}{2}\varphi-\left(\frac{x}{a}+\frac{y}{b}+1\right)\tan^2\tfrac{1}{2}\varphi=0.$$

$\therefore$ The equation to the envelope is